

Due Sun

lambda
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5.1 – Eigenvalues and Eigenvectors

Consider the equation $\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$. Form: $A\vec{x} = \lambda \vec{x}$

Definition: If A is an $n \times n$ matrix, then a nonzero vector \mathbf{x} in R^n is called an **eigenvector** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; that is, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . The scalar λ is called an **eigenvalue** of A (or of T_A), and \mathbf{x} is said to be an **eigenvector corresponding to λ** .

In the above equation, 5 is an eigenvalue of the matrix A , and $\begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$ is its associated eigenvector.

Note that if $A\vec{x} = \lambda\vec{x}$, $\lambda\vec{x} - A\vec{x} = \vec{0}$. Then $(\lambda I - A)\vec{x} = \vec{0}$. Since $\vec{x} \neq \vec{0}$ we seek nontrivial solutions

Theorem 5.1.1 If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation $\det(\lambda I - A) = 0$.

If A is an $n \times n$ matrix, then the **characteristic equation** of A is $\det(\lambda I - A) = 0$.

Definition: The **characteristic polynomial** of an $n \times n$ matrix A results from expanding the characteristic equation of A and has the form

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n.$$

The solution space of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ .

#6 In each part, find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

b. $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$

a) $\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{bmatrix}$

$\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = 0$

$\lambda^2 - 4\lambda + 4 - 1 = 0 \Rightarrow (\lambda - 3)(\lambda - 1) = 0$ *characteristic equation*

OR $(\lambda - 2)^2 - 1 = 0 \Rightarrow (\lambda - 2)^2 = 1 \Rightarrow \lambda - 2 = \pm 1$

Eigenvalues: $\lambda_1 = 3, \lambda_2 = 1$

$\lambda_1 = 3: \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 = x_2 \Rightarrow \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 1: \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *eigenvector*

Basis for eigenspace for $\lambda_1 = 3$ is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

For $\lambda_2 = 1$, it's $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

algebraic multiplicity 2

b) $\det(\lambda I - A) = 0 \Rightarrow \begin{vmatrix} \lambda - 2 & 3 \\ 0 & \lambda - 2 \end{vmatrix} = 0 \Rightarrow (\lambda - 2)^2 = 0$

$\lambda = 2 \rightarrow \begin{bmatrix} 0 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow x_2 = 0, x_1 \text{ is free}$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \text{basis for eigenspace: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$$c) \begin{vmatrix} \lambda-2 & 0 \\ 0 & \lambda-2 \end{vmatrix} = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \lambda = 2$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Basis: } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}.$$

$$d) \begin{vmatrix} \lambda-1 & 2 \\ -2 & \lambda+1 \end{vmatrix} = 0 \Rightarrow \lambda^2 - 1 + 4 = 0 \Rightarrow \lambda^2 = -3$$

This has no real eigenvalues.

Theorem 5.1.3 If A is an $n \times n$ matrix, the following statements are equivalent.

- λ is an eigenvalue of A .
- λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
- The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
- There is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$.

#9 find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \quad \left| \begin{array}{ccc} \lambda-6 & -3 & 8 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda+3 \end{array} \right| = 0$$

$$(\lambda+2) \left| \begin{array}{cc} \lambda-6 & 8 \\ -1 & \lambda+3 \end{array} \right| = 0 \Rightarrow (\lambda+2)(\lambda^2 - 3\lambda - 18 + 8) = 0$$

$$\Rightarrow (\lambda+2)(\lambda-5) = 0, \lambda = -2, 5$$

$\lambda = -2$ is an eigenvalue of alg. mult. 2

$$\lambda = -2 \quad \begin{bmatrix} -8 & -3 & 8 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = x_3 \\ x_2 = 0 \end{array}$$

$\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is ~~an~~ the eigenvector associated with $\lambda = -2$. $\lambda = -2$ has geom. mult. 1.

$$\lambda = 5 \begin{bmatrix} -1 & -3 & 8 \\ 0 & 7 & 0 \\ -1 & 0 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{cases} x_1 = 8x_3 \\ x_2 = 0 \end{cases}$$

eigenvector: $\vec{x} = \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}$

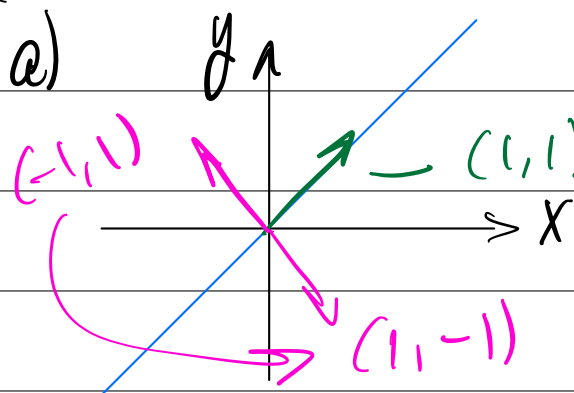
Eigenvalues & bases for eigenspaces:

$$\lambda_1 = -2, \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}; \lambda_2 = 5, \left\{ \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

#19 In each part, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on \mathbb{R}^2 . Use geometric reasoning to find the answers. No computations are needed.

- Reflection about the line $y = x$.
- Orthogonal projection onto the x -axis.
- Rotation about the origin through a positive angle of 90° .
- Contraction with factor k ($0 \leq k < 1$)
- Shear in the x -direction by a factor k ($k \neq 0$)

$$A\vec{x} = \lambda\vec{x}$$

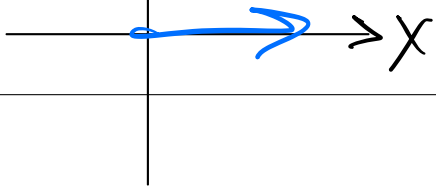


$$\vec{x} = \begin{bmatrix} a \\ a \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} a \\ -a \end{bmatrix}$$

$\lambda = 1$ $\lambda = -1$

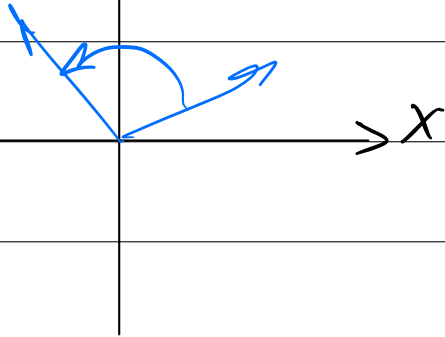
Eigenspace:
 $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $\text{Span} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

b) y λ



$$\lambda = 1, \text{ span } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

c) y λ



$$A\vec{x} = \lambda\vec{x} \text{ has}$$

no solution unless

$$\theta = n\pi$$

this has no real eigenvalues for arbitrary angles θ .

Theorem 5.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A .

#14 Find the characteristic equation of the matrix by inspection.

$$\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

$$(\lambda - 9)(\lambda + 1)(\lambda - 3)(\lambda - 7) = 0$$

#15 Find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula.

$$T(x, y) = (x + 4y, 2x + 3y) \rightarrow \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \rightarrow \text{Proceed as above}$$

$\lambda = 0$ is a legit eigenvalue

$\vec{x} = \vec{0}$ is not allowed as an eigenvector (by def)

Theorem 5.1.4 A square matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A .

Recall λ is an eigenvalue of A if

\exists a nonzero vector $\vec{x} \ni A\vec{x} = \lambda\vec{x}$.

Then $\det(\lambda I - A) = 0$.

(\Rightarrow) Suppose A is invertible.

$$\begin{array}{c} n \times n \\ \downarrow \\ \det(kA) = k^n \det(A) \end{array}$$

Consider $\det(A) = 0 \Rightarrow \det(-A) = 0$

If $\lambda = 0$ is an eigenvalue, then

$$\det(\lambda I - A) = 0 \Rightarrow \det(-A) = 0$$

So $\lambda = 0$ is not an eigenvalue of A .

(\Leftarrow) If $\lambda = 0$ is not an eigenvalue of A , then there is no nonzero vector \vec{x}

satisfying $A\vec{x} = 0 \vec{x} = \vec{0} \Rightarrow$ the system
has only the trivial solution
 $\Rightarrow A$ is invertible.

Theorem 5.1.5 Equivalent Statements (extends Theorem 4.9.8)

If A is an $n \times n$ matrix, then the following statements are equivalent.

a) A is invertible.

b) $Ax = \mathbf{0}$ has only the trivial solution.

c) The reduced row echelon form of A is I_n .

d) A is expressible as a product of elementary matrices.

e) $Ax = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} .

f) $Ax = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} .

g) $\det(A) \neq 0$.

h) The column vectors of A are distinct and linearly independent.

i) The row vectors of A are distinct and linearly independent.

j) The column vectors of A span R^n .

k) The row vectors of A span R^n .

l) The column vectors of A form a basis for R^n .

m) The row vectors of A form a basis for R^n .

n) A has rank n .

o) A has nullity 0.

p) The orthogonal complement of the null space of A is R^n .

q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.

r) $\lambda = 0$ is not an eigenvalue of A . ✓